

PROCEEDINGS

OF

THE ROYAL SOCIETY

~~~~~

*January 18, 1894.*

The LORD KELVIN, D.C.L., LL.D., President, followed by Sir JOHN EVANS, K.C.B., D.C.L., LL.D., Vice-President and Treasurer, in the Chair.

The Right Hon. James Bryce was admitted into the Society.

A List of the Presents received was laid on the table, and thanks ordered for them.

The following Papers were read :—

I. "On Homogeneous Division of Space." By LORD KELVIN,  
P.R.S. Received January 17, 1894.

§ 1. The homogeneous division of any volume of space means the dividing of it into equal and similar parts, or cells, as I shall call them, all sameways oriented. If we take any point in the interior of one cell or on its boundary, and corresponding points of all the other cells, these points form a homogeneous assemblage of single points, according to Bravais' admirable and important definition.\* The general problem of the homogeneous partition of space may be stated thus :— Given a homogeneous assemblage of single points, it is required to find every possible form of cell enclosing each of them subject to the condition that it is of the same shape and sameways oriented for all. An interesting application of this problem is to find for a crystal (that is to say, a homogeneous assemblage of groups of chemical atoms) a homogeneous arrangement of partitional interfaces such that each cell contains all the atoms of one molecule. Unless we

\* 'Journal de l'École Polytechnique,' tome 19, cahier 33, pp. 1—128 (Paris, 1850), quoted and used in my 'Mathematical and Physical Papers,' vol. 3, art. 97, p. 400.

knew the exact geometrical configuration of the constituent parts of the group of atoms in the crystal, or crystalline molecule as we shall call it, we could not describe the partitional interfaces between one molecule and its neighbour.

Knowing as we do know for many crystals the exact geometrical character of the Bravais assemblage of corresponding points of its molecules, we could not be sure that any solution of the partitional problem we might choose to take would give a cell containing only the constituent parts of one molecule. For instance, in the case of quartz, of which the crystalline molecule is probably  $3(\text{SiO}_2)$ , a form of cell chosen at random might be such that it would enclose the silicon of one molecule with only some part of the oxygen belonging to it, and some of the oxygen belonging to a neighbouring molecule, leaving out some of its own oxygen, which would be enclosed in the cell of either that neighbour or of another neighbour or other neighbours.

§ 2. This will be better understood if we consider another illustration—a homogeneous assemblage of equal and similar trees planted close together in any regular geometrical order on a plane field either inclined or horizontal, so close together that roots of different trees interpenetrate in the ground, and branches and leaves in the air. To be perfectly homogeneous, every root, every twig, and every leaf of any one tree must have equal and similar counterparts in every other tree. So far everything is natural, except, of course, the absolute homogeneity that our problem assumes; but now, to make a homogeneous assemblage of molecules in space, we must suppose plane above plane each homogeneously planted with trees at equal successive intervals of height. The interval between two planes may be so large as to allow a clear space above the highest plane of leaves of one plantation and below the lowest plane of the ends of roots in the plantation above. We shall not, however, limit ourselves to this case, and we shall suppose generally that leaves of one plantation intermingle with roots of the plantation above, always, however, subject to the condition of perfect homogeneity. Here, then, we have a truly wonderful problem of geometry—to enclose ideally each tree within a closed surface containing every twig, leaf, and rootlet belonging to it, and nothing belonging to any other tree, and to shape this surface so that it will coincide all round with portions of similar surfaces around neighbouring trees. Wonderful as it is, this is a perfectly easy problem if the trees are given, and if they fulfil the condition of being perfectly homogeneous.

In fact we may begin with the actual bounding surface of leaves, bark, and roots of each tree. Wherever there is a contact, whether with leaves, bark, or roots of neighbouring trees, the areas of contact form part of the required cell-surface. To complete the cell-surface we

have only to swell out\* from the untouched portions of surface of each tree homogeneously until the swelling portions of surface meet in the interstitial air spaces (for simplicity we are supposing the earth removed, and roots, as well as leaves and twigs, to be perfectly rigid). The wonderful cell-surface which we thus find is essentially a case of the tetrakaidekahedronal cell, which I shall now describe for any possible homogeneous assemblage of points or molecules.

§ 3. We shall find that the form of cell essentially consists of fourteen walls, plane or not plane, generally not plane, of which eight are hexagonal and six quadrilateral; and with thirty-six edges, generally curves, of meeting between the walls; and twenty-four corners where three walls meet. A cell answering this description must of course be called a tetrakaidekahedron, unless we prefer to call it a fourteen-walled cell. Each wall is an interface between one cell and one of fourteen neighbours. Each of the thirty-six edges is a line common to three neighbours. Each of the twenty-four corners is a point common to four neighbours. The old-known parallelepipedal partitioning is merely a very special case in which there are four neighbours along every edge, and eight neighbours having a point in common at every corner. We shall see how to pass (§ 4) continuously from or to this singular case, to or from a tetrakaidekahedron differing infinitesimally from it; and, still continuously, to or from any or every possible tetrakaidekahedronal partitioning.

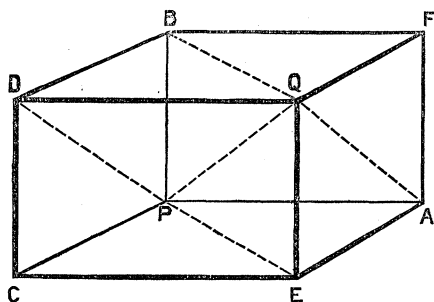
§ 4. To change from a parallelepipedal to a tetrakaidekahedronal cell, for one and the same homogeneous distribution of points, proceed thus:—Choose any one of the four body-diagonals of a parallelepiped and divide the parallelepiped into six tetrahedrons by three planes each through this diagonal, and one of the three pairs of parallel edges which intersect it in its two ends. Give now any purely translational motion to each of these six tetrahedrons. We have now the  $4 \times 6$  corners of these tetrahedrons at twenty-four distinct points. These are the corners of a tetrakaidekahedron, such as that described generally in § 3. The two sets of six corners, which before the movement coincided in the two ends of the chosen diagonal, are now the corners of one pair of the hexagonal faces of the tetrakaidekahedron. When we look at the other twelve corners we see them as corners of other six hexagons, and of six parallelograms, grouped together as described in § 15 below. The movements of the six tetrahedrons may be such that the groups of six corners and of four corners are in fourteen planes as we shall see in § 14; but, if they are made at random, none of the groups will be in a single plane. The fourteen faces, plane or not plane, of the tetrakaidekahedron are obtained by drawing arbitrarily any set of surfaces to constitute four of the hexagons and three of the quadrilaterals, with arbitrary curves for the edges between hexagon and

\* Compare 'Mathematical and Physical Papers,' vol. 3, art. 97, § 5.

hexagon and between hexagons and quadrilaterals, and then by drawing parallel equal and similar counterparts to these surfaces in the remaining four hexagonal and three quadrilateral spaces in the manner more particularly explained in § 6 below. It is clear, or at all events I shall endeavour to make it clear by fuller explanations and illustrations below, that the figure thus constituted fulfils our definition (§1) of the most general form of cell fitted to the particular homogeneous assemblage of points corresponding to the parallelepiped with which we have commenced. This will be more easily understood in general, if we first consider the particular case of *parallelepipedal* partitioning, and of the deviations which, without altering its corners, we may arbitrarily make from a plane-faced parallelepiped, or which we may be compelled by the particular figure of the molecule to make.

§ 5. Consider, for example, one of the trees of § 2, or if you please a solid of less complex shape, which for brevity we shall call S, being one of a homogeneous assemblage. Let P be a point in unoccupied space (air, we shall call it for brevity), which, for simplicity we may suppose to be somewhere in the immediate neighbourhood of S, although it might really be anywhere far off among distant solids of the assemblage. Let PA, PB, PC be lines parallel to any three Bravais rows not in one plane, and let A, B, C be the nearest points corresponding to P in these lines. Complete a parallelepiped on the lines PA, PB, PC, and let QD, QE, QF be the edges parallel to them

(FIG. 7, OF § 9.)



through the opposite corner Q. Because of the homogeneousness of the assemblage, and because A, B, C, D, E, F, Q are points corresponding to P, which is in air, each of those seven points is also in air. Draw any line through air from P to A and draw the lines of corresponding points from B to F, D to Q, and C to E. Do the same relatively to PB, AF, EQ, CD; and again the same relatively to

PC, AE, FQ, BD. These twelve lines are all in air, and they are the edges of our curved-faced parallelepiped. To describe its faces take points infinitely near to one another along the line PC (straight or curved as may be): and take the corresponding points in BD. Join these pairs of corresponding points by lines in air infinitely near to one another in succession. These lines give us the face PBDC. Corresponding points in AE, FQ, and corresponding lines between them give us the parallel face AFQE. Similarly we find the other two pairs of the parallel faces of the parallelepiped. If the solids touch one another anywhere, either at points or throughout finite areas, we are to reckon the interface between them as air in respect to our present rules.

§ 6. We have thus found the most general possible parallelepipedal partitioning for any given homogeneous assemblage of solids. Precisely similar rules give the corresponding result for *any possible partitioning* if we first choose the twenty-four corners of the tetrakaidekahedron by finding six tetrahedrons and giving them arbitrary translatory motions according to the rule of § 4. To make this clear it is only now necessary to remark that the four corners of each tetrahedron are essentially corresponding points, and that if one of them is in air all of them are in air, whatever translatory motion we give to the tetrahedron.

§ 7. The transition from the parallelepiped to the tetrakaidekahedron described in § 4 will be now readily understood, if we pause to consider the vastly simpler two-dimensional case of transition from a parallelogram to a hexagon. This is illustrated in figs. 1 and 2; with heavy lines in each case for the sides of the hexagon, and light lines for the six of its diagonals which are sides of constructional triangles. The four diagrams show different relative positions in one plane of two equal homochirally similar triangles ABC, A'B'C'; oppositely

FIG. 1.

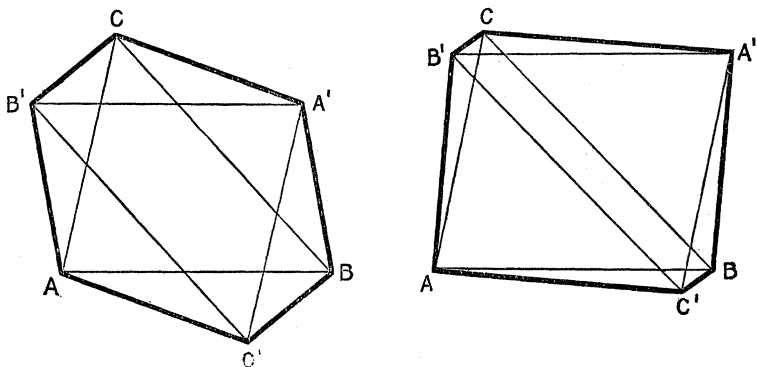
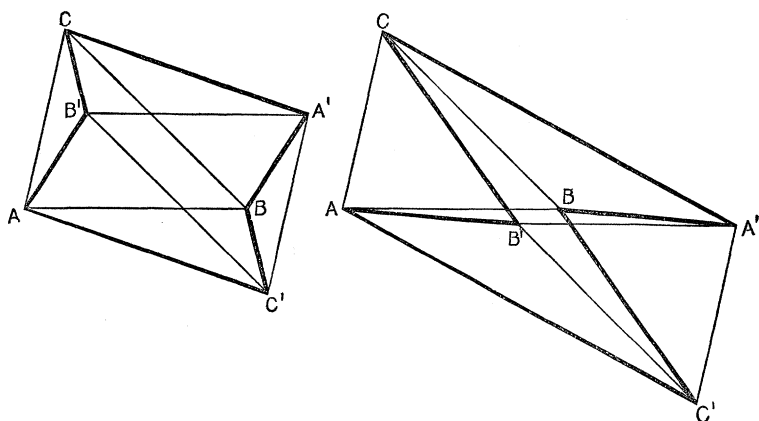


FIG. 2.



oriented (that is to say, with corresponding lines  $AB$ ,  $A'B'$  parallel but in inverted directions). The hexagon  $AC'BA'CB'$ , obtained by joining  $A$  with  $B'$  and  $C'$ ,  $B$  with  $C'$  and  $A'$ , and  $C$  with  $A'$  and  $B'$ , is clearly in each case a proper cell-figure for dividing plane space homogeneously according to the Bravais distribution of points defined by either triangle, or by putting the triangles together in any one of the three proper ways to make a parallelogram of them. The corresponding operation for three-dimensional space is described in § 4: and the proof which is obvious in two-dimensional space is clearly valid for space of three dimensions, and therefore the many words which would be required to give it formal demonstration are superfluous.

§ 8. The principle according to which we take arbitrary curved surfaces with arbitrary curved edges of intersection, for seven of the faces of our partitional tetrakaidekahedron, and the other seven correspondingly parallel to them, is illustrated in figs. 3, 4, 5, and 6, where the corresponding thing is done for a partitional hexagon suited to the homogeneous division of a plane. In these diagrams the hexagon is for simplicity taken equilateral and equiangular. In drawing fig. 3, three pieces of paper were cut, to the shapes  $kl$ ,  $mn$ ,  $uv$ . The piece  $kl$  was first placed in the position shown relatively to  $AC'$ , and a portion of the area of one cell to be given to a neighbour across the frontier  $C'A$  on one side was marked off. It was then placed in the position shown relatively to  $A'C$  and the equivalent portion to be taken from a neighbour on the other side was marked. Corresponding give-and-take delimitations were marked on the frontiers  $C'B$  and  $B'C$ , according to the form  $mn$ ; and on the frontiers  $BA'$ ,  $AB'$ , according to the form  $uv$ . Fig. 4 was drawn on the same plan

FIG. 3.

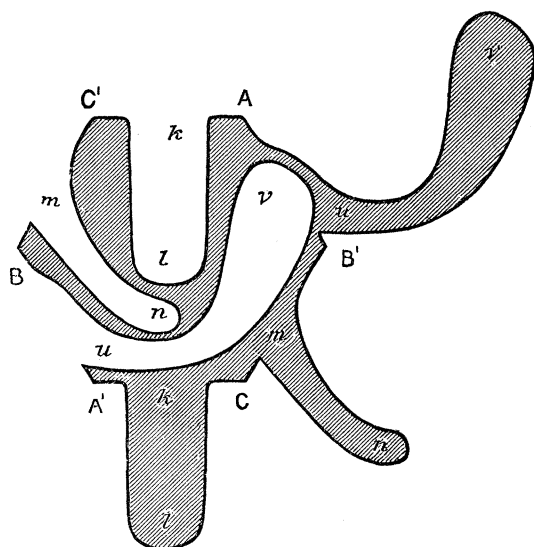
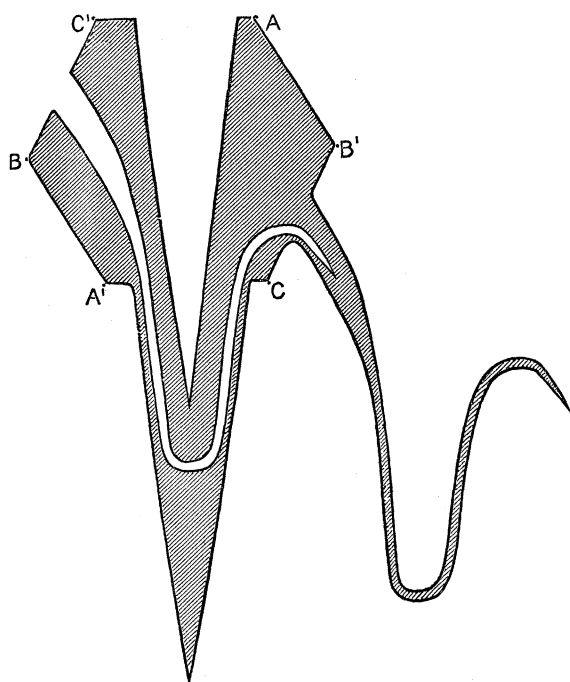


FIG. 4.



but with one pair of frontiers left as straight lines, and the two other pairs drawn by aid of two paper templets. It would be easy, but not worth the trouble, to cut out a large number of pieces of brass of the shapes shown in these diagrams and to show them fitted together like the pieces of a dissected map. Figs. 5 and 6 are drawn on the same principle; fig. 6 showing, on a reduced scale, the result of putting pieces together precisely equal and similar to that shown in

FIG. 5.

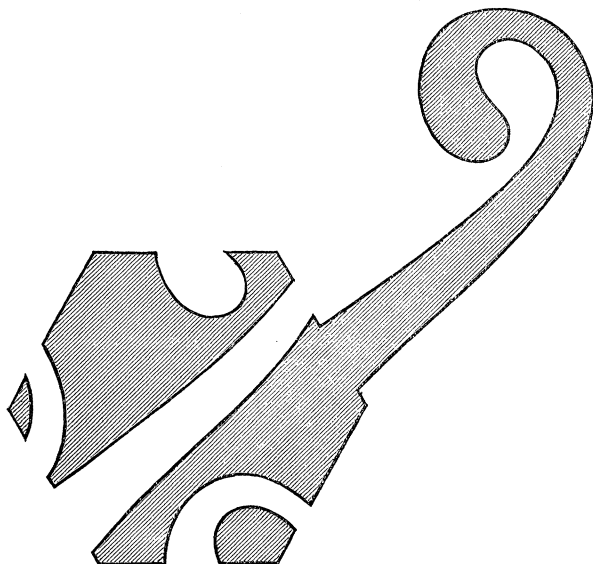
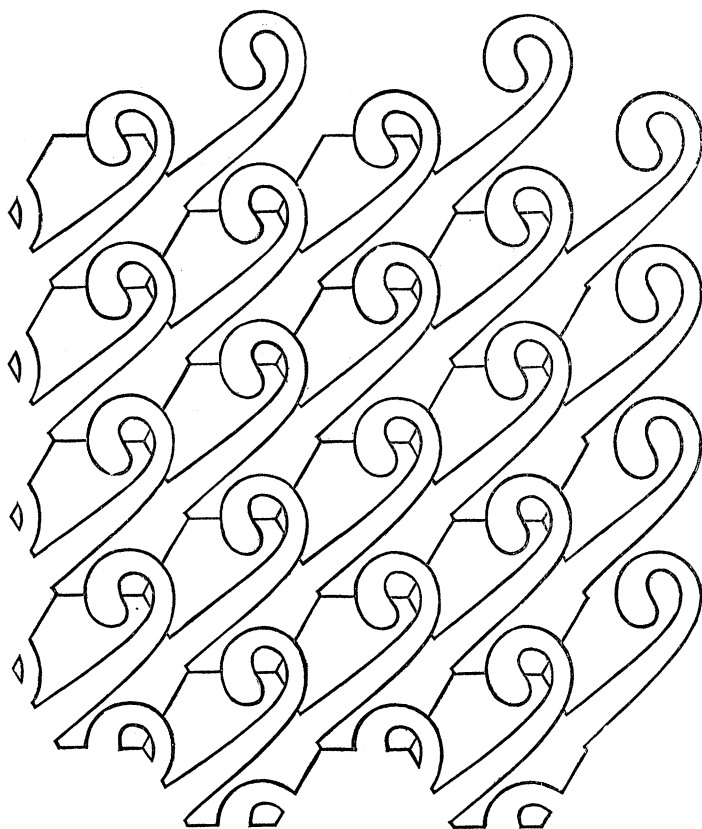


fig. 5. In these diagrams, unlike the cases represented in figs. 3 and 4, the primitive hexagon is, as shown clearly in fig. 5, divided into isolated parts. But if we are dealing with homogeneous division of solid space, the separating channels shown in fig. 5 might be sections, by the plane of the drawing, of perforations through the matter of one cell produced by the penetration of matter, rootlets for example, from neighbouring cells.

§ 9. Corresponding to the three ways by which two triangles can be put together to make a parallelogram, there are seven, and only seven, ways in which the six tetrahedrons of § 4 can be put together to make a parallelepiped, in positions parallel to those which they had in the original parallelepiped. To see this, remark first that among the thirty-six edges of the six tetrahedrons seven different lengths are found which are respectively equal to the three lengths of edges (three quartets of equal parallels); the three

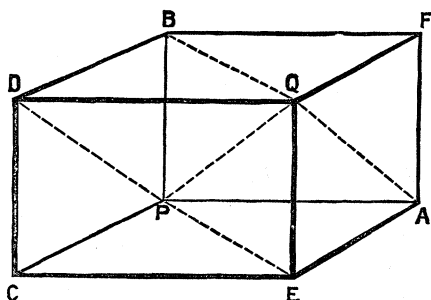
FIG. 6.



lengths of face-diagonals having ends in P or Q (three pairs of equal parallels); and the length of the chosen body-diagonal PQ. (Any one of these seven is, of course, determinable from the other six if given.)

In the diagram, fig. 7, full lines show the edges of the primitive parallelepiped, and dotted lines show the body-diagonal PQ and two pairs of the face-diagonals, the other pair of face-diagonals (PF, QC), not being marked on the diagram to avoid confusion. Thus, the diagram shows, in the parallelograms QDPA and QEPB, two of the three cutting planes by which it is divided into six tetrahedrons, and it so shows also two of the six tetrahedrons, QPDB and QPEA. The lengths QP, QD, QE, QF are found in the edges of every one of the six tetrahedrons, the two other edges of each being of two of the three lengths QA, QB, QC. The six tetrahedrons may be taken in order of

FIG. 7.



three pairs having edges of lengths respectively equal to  $QB$  and  $QC$ ,  $QC$  and  $QA$ ,  $QA$  and  $QB$ . It is the third of these pairs that is shown in fig. 7. Remark now that the sum of the six angles of the six tetrahedrons at the edge equal to any one of the lengths  $QP$ ,  $QD$ ,  $QE$ ,  $QF$  is four right angles. Remark also that the sum of the four angles at the edge of length  $QA$  in the two pairs of tetrahedrons in which the length  $QA$  is found is four right angles, and the same with reference to  $QB$  and  $QC$ . Remark lastly that the two tetrahedrons of each pair are equal and dichirally\* similar, or enantiomorphs as such figures have been called by German writers.

§ 10. Now, suppose any one pair of the tetrahedrons to be taken away from their positions in the primitive parallelepiped, and, by purely translational motion, to be brought into position with their edges of length  $QD$  coincident, and the same to be done for each of the other two pairs. The sum of the six angles at the coincident edges being two right angles, the plane faces at the common edge will fit together, and the condition of parallelism in the motion of each pair fixes the order in which the three pairs come together in the new position, and shows us that in this position the three pairs form a parallelepiped essentially different from the primitive parallelepiped, provided that, for simplicity in our present considerations, we suppose each tetrahedron to be wholly scalene, that is to say, the seven lengths found amongst the edges to be all unequal. Next shift the tetrahedrons to bring the edges  $QE$  into coincidence, and next again to bring the edges  $QF$  into coincidence. Thus, including the primitive parallelepiped, we can make four different parallelepipeds in each of which six of the tetrahedrons have a common edge.

§ 11. Now take the two pairs of tetrahedrons having edges of length equal to  $QA$ , and put them together with these edges coincident. Thus we have a scalene octahedron. The remaining pair of

\* A pair of gloves are dichirally similar, or enantiomorphs. Equal and similar right-handed gloves are chirally similar.

tetrahedrons placed on a pair of its parallel faces complete a parallelepiped. Similarly two other parallelepipeds may be made by putting together the pairs that have edges of lengths equal to QB and QC respectively with those edges coincident, and finishing in each case with the remaining pair of tetrahedrons. The three parallelepipeds thus found are essentially different from one another, and from the four of § 10; and thus we have the seven parallelepipeds fulfilling the statement of § 9. Each of the seven parallelepipeds corresponds to one and the same homogeneous distribution of points.

§ 12. Going back to § 4, we see that, by the rule there given, we find four different ways of passing to the tetrakaidekahedron from any one chosen parallelepiped of a homogeneous assemblage. The four different cellular systems thus found involve four different sets of seven pairs of neighbours for each point. In each of these there are four pairs of neighbours in rows parallel to the three quartets of edges of the parallelepiped and to the chosen body-diagonal; and the other three pairs of neighbours are in three rows parallel to the face-diagonals which meet in the chosen body-diagonal. The second (§ 11) of the two modes of putting together tetrahedrons to form a parallelepiped which we have been considering suggests a second mode of dividing our primitive parallelepiped, in which we should first truncate two opposite corners and then divide the octahedron which is left, by two planes through one or other of its three diagonals. The six tetrahedrons obtained by any one of the twelve ways of effecting this second mode of division give, by their twenty-four corners, the twenty-four corners of a space-filling tetrakaidekahedronal cell, by which our fundamental problem is solved. But every solution thus obtainable is clearly obtainable by the simpler rule of § 4, commencing with some one of the infinite number of primitive parallelepipeds which we may take as representative of any homogeneous distribution of points.

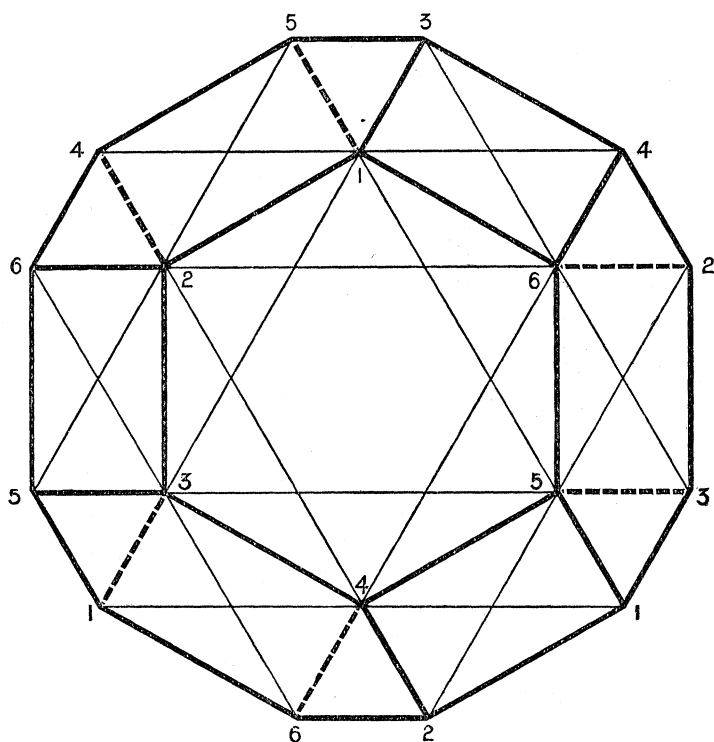
§ 13. The communication is illustrated by a model showing the six tetrahedrons derived by the rule 4 from a symmetrical kind of primitive parallelepiped, being a rhombohedron of which the axial-diagonal is equal in length to each of the edges. The homogeneous distribution of points corresponding to this form of parallelepiped is the well-known one in which every point is surrounded by eight others at the corners of a cube of which it is the centre; or, if we like to look at it so, two simple cubical distributions of single points, each point of one distribution being at the centre of a cube of points of the other. [To understand the tactics of the single homogeneous assemblage constituted by these two cubic assemblages, let P be a point of one of the cubic assemblages, and Q any one of its four nearest neighbours of the other assemblage. Q is at the centre of a cube of which P is at one corner. Let PD, PE, PF be three conterminous edges of this cube so

that A, B, C are points of the first assemblage nearest to P. Again Q is a corner of a cube of which P is the centre; and if QA, QB, QC are three conterminous edges of this cube, D, E, F are points of the second assemblage nearest to Q. The rhombohedron of which PQ is body-diagonal and PA, PB, PC the edges conterminous in P, and QD, QE, QF the edges conterminous in Q, is our present rhombohedron. The diagram of § 9 (fig. 7), imagined to be altered to proper proportions for the present case, may be looked to for illustration. Its three face-diagonals through P, being PD, PE, PF, are perpendicular to one another. So also are QA, QB, QC, its three face-diagonals through Q. The body-diagonal of the cube PQ, being half the body-diagonal of the cube whose edges are PD, PE, PF, is equal to  $PD \times \sqrt{\frac{3}{2}}$ : and PA, PB, PC are also each of them equal to this, because A, B, C are centres of other equal cubes, having P for a common corner.—January 30.]

§ 14. The tetrahedrons used in the model are those into which the parallelepiped is cut by three planes through the axial diagonal, which in this case cut one another at angles of  $60^\circ$ . We wish to be able to shift the tetrahedrons into positions corresponding to those of the triangles in fig. 1, which we could not do if they were cut out of the solid. I, therefore, make a mere skeleton of each tetrahedron, consisting of a piece of wire bent at two points, one-third of its length from its ends, at angles of  $70\frac{1}{2}^\circ$ , being  $\sin^{-1} \frac{1}{3} \sqrt{3}$ , in planes inclined at  $60^\circ$  to one another. The six skeletons thus made are equal and similar, three homochirals and the other three also homochirals, their enantiomorphs. In their places in the primitive parallelepiped they have their middle lines coincident in its axial diagonal PQ, and their other  $6 \times 2$  arms coincident in three pairs in its six edges through P and Q. Looking at fig. 7 we see, for example, three of the edges CP, PQ, QE, of one of the tetrahedrons thus constituted; and DQ, QP, PB, three edges of its enantiomorph. In the model they are put together with their middle lines at equal distances around the axial diagonal and their arms symmetrically arranged round it. Wherever two lines cross they are tied, not very tightly, together by thin cord many times round, and thus we can slip them along so as to bring the six middle lines either very close together, nearly as they would be in the primitive parallelepiped, or farther and farther out from one another so as to give, by the four corners of the tetrahedrons, the twenty-four corners of all possible configurations of the plane-faced space-filling tetrakaidekahedron.

§ 15. The six skeletons being symmetrically arranged around an axial line we see that each arm is cut by lines of other skeletons in three points. For an important configuration, let the skeletons be separated out from the axial line just so far that each arm is divided into four equal parts, by those three intersectional points. The

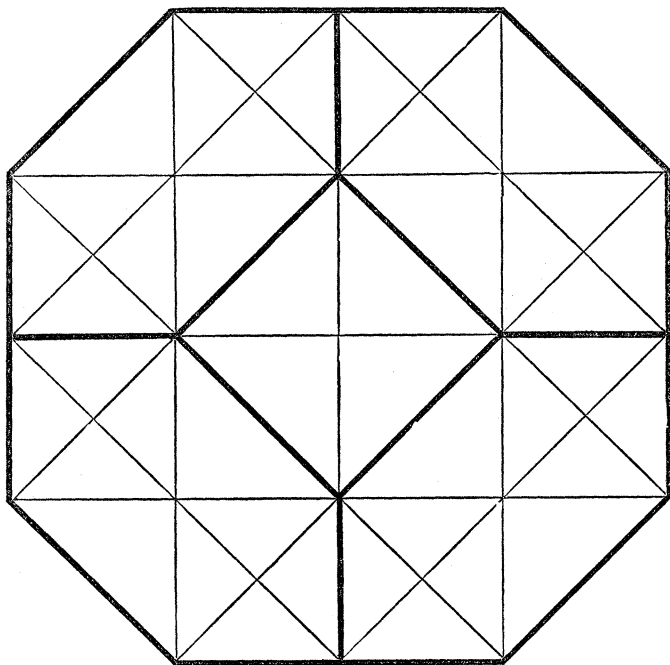
FIG. 8.



tetrakaidekahedron of which the twenty-four corners are the corners of the tetrahedrons thus placed may conveniently be called the orthic tetrakaidekahedron. It has six equal square faces and eight equal equiangular and equilateral hexagonal faces. It was described in § 12 of my paper on "The Division of Space with Minimum Partitional Area,"\* under the name of "plane-faced isotropic tetrakaidekahedron"; but I now prefer to call it orthic, because, for each of its seven pairs of parallel faces, lines forming corresponding points in the two faces are perpendicular to the faces, and the planes of its three pairs of square faces are perpendicular to one another. Fig. 8 represents an orthogonal projection on a plane parallel to one of the four pairs of hexagonal faces. The heavy lines are edges of the tetrakaidekahedron. The light lines are edges of the tetrahedrons of § 13, or parts of those edges not coincident in projection with the edges of the tetrakaidekahedron. The figures 1, 1, 1; 2, 2, 2; . . .; 6, 6, 6 show corners belonging respectively to the six tetrahedrons,

\* 'Phil. Mag.,' 1887, 2nd half-year, and 'Acta Mathematica,' vol. 11, pp. 121—134.

FIG. 9.



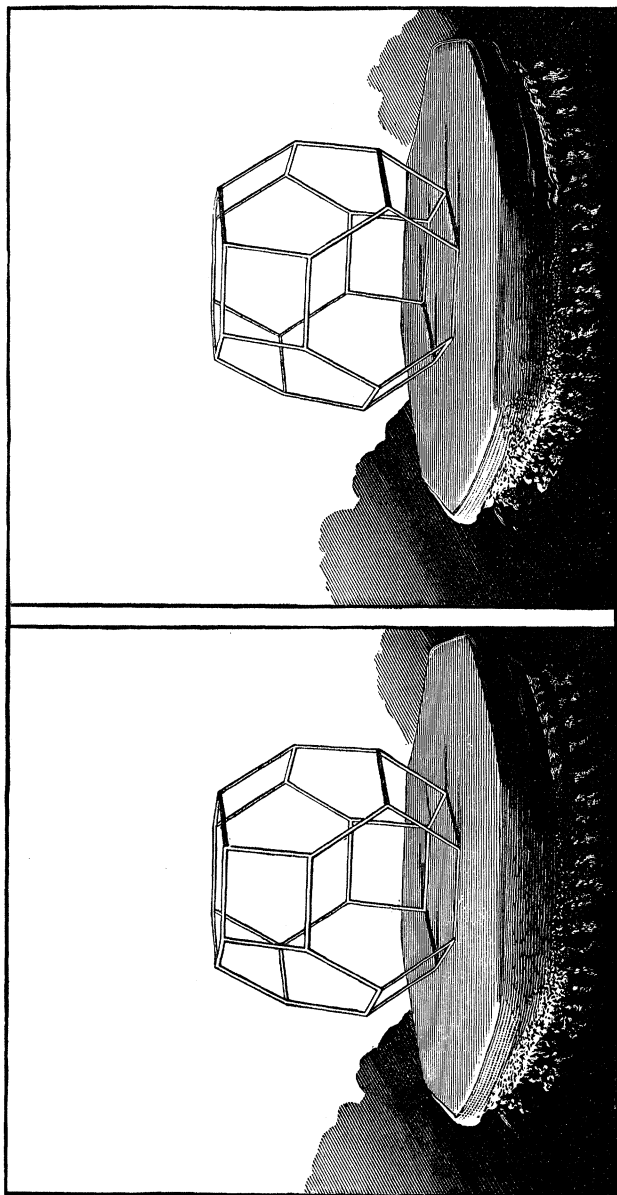
two of the four corners of each being projected on one point in the diagram. Fig. 9 shows, on the same scale of magnitude with corresponding distinction between heavy and light lines, the orthogonal projection on a plane parallel to a pair of square faces.

§ 16. If the rule of § 15 with reference to the division of each arm of a skeleton tetrahedron into four equal parts by points in which it is cut by other lines of skeletons is fulfilled with all details of §§ 14 and 15 applied to any oblique parallelepiped, we find a tetrakaidekahedron which we may call orthoid, because it is an orthic tetrakaidekahedron, altered by homogeneous strain. Professor Crum Brown has kindly made for me the beautiful model of an orthoidal tetrakaidekahedron thus defined which is placed before the Royal Society as an illustration of the present communication.

Fig. 10 is a stereoscopic picture of an orthic tetrakaidekahedron, made by soldering together thirty-six pieces of wire, each 4 in. long, with three ends of wire at each of twenty-four corners.

§ 17. I cannot in the present communication enter upon the most general possible plane-faced partitional tetrakaidekahedron or show its relation to orthic and orthoidal tetrakaidekahedrons. I may

Fig. 10.



merely say that the analogy in the homogeneous division of a plane is this:—an equilateral and equiangular hexagon (orthic); any other hexagon of three pairs of equal and parallel sides whose paracentric

diagonals trisect one another (orthoidal). The angles of an orthoidal hexagon, other than equilateral, are not  $120^\circ$ . The angles of the left-hand hexagon fig. 1 (§ 7) are  $120^\circ$ , and its paracentric diagonals do not trisect one another, as the diagram clearly shows.

II. "An Estimate of the Degree of Legitimate Natality, as shown in the Table of Natality compiled by the Author from Observations made at Budapest." By JOSEPH KŐRÖSI, Member of the Hungarian Academy of Sciences, Director of Municipal Statistics. Communicated by Sir JAMES PAGET, F.R.S. Received December 28, 1893.

(Abstract.)

Both branches of the science of demography—natality statistics as well as mortality statistics—originated on British soil. It was in 1665 that the Royal Society published the first work on these matters (Graunt's "Observations"), whilst in 1693 Halley, by establishing the first life table, laid the foundation of the scientific treatment of mortality statistics. These tables of mortality showed for the first time how to measure the probability of *death* for each year of human age. The other branch of vital statistics is still in want of a corresponding table of natality, showing the probability of *birth* for each of the age-combinations of the parents. The table of natality is not of so great scientific importance as the life table, as the probability of death depends on natural laws, whilst the fertility, at least partially, is influenced by voluntary causes also. But as the problems of over-population or de-population are an effect of both forces, it is worth while to study the law of these facts also.

To reach this aim, I have tabulated the age of the 71,800 married couples given in the Census of 1891, conforming to the single year-combinations. The virtual number of these combinations—as 45 productive years of the male have to be combined with each of the 40 productive years of the female—is about 2000. Knowing thus the number of all age-combinations, I observed for four years (two before and two after the Census) the 46,931 births amongst couples of those ages; I got thus, dividing the figures obtained by four, the yearly probability of birth for each age-combination.

As the legitimate natality is to be regarded as a resultant between two distinct forces, the instinct of nature which urges towards multiplication and the forethought which causes moral restraint, it was also desirable to get an insight into the march of the physiological fertility alone. For this purpose I had to look out for couples in whom the moral restraint is weakest or entirely absent. These con-

FIG. 10.

